Cavity QED with Ultracold Gases

Ultracold Gases  Cavity QED

Optomechanics

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Basic Theme and Physics

**Quantum optics:**
quantized light modes + fixed classical matter

**Ultracold gases:**
quantum particle motion + fixed classical optical potential

Quantum optics with quantum gases:
full quantum description of light and matter waves

*light induces a*
*dynamic optical lattice with quantized depth*

*atoms generate a*
*dynamic refractive index with quantum properties*

H. R., P Domokos, F. Brennecke, T. Esslinger, Rev. Mod. Phys., 2013
• **Cavity QED and Cavity Cooling**

• **Ground state cooling and atomic quantum statistics in ( multimode ) resonators**

• **Quantum dynamics of selfordering in multimode cavities**
Light force basics

Two classes of forces on point particles (CCT, JD):

**Radiation pressure**
(absorption + spontaneous re-emission)

**Dipole force**
(absorption + stimulated emission)

\[ \vec{F}_{rad} = -\hbar \Omega v_0 \left( \frac{\Gamma^2}{4} + \frac{1}{2} \Omega^2 \right) \]

\[ \vec{F}_{dip} = \hbar (\omega_l - \omega_a) \frac{1}{(\omega_l - \omega_a)^2 + \Gamma^2/4 + \frac{1}{2} \Omega^2} \]

- **negligible in dispersive large detuning limit!**
- **coherent transfer of momentum, which depends on relative phases of fields**

- **conservative optical potential ~ intensity**
- **scales as 1/detuning**
- **optical tweezers, (mirror reflection = called “radiation pressure”)**
Lightforces on moving polarizable particles in optical resonators

**Light forces of resonator field determine atomic motion**

- self - trapping
- friction + diffusion
- correlation and entanglement

Particle position and motion changes resonator field dynamics

**dispersive regime**

(= dipol-force) at large laser-atom detuning:

\[
\Delta = \omega_{\text{laser}} - \omega_{\text{atom}} > \gamma, \kappa
\]

\[
\omega_{\text{laser}} \sim \omega_{\text{cavity}}
\]

dipole force dominates spontaneous emission
**Dispersive Cavity QED:**

*at large atom-cavity detuning $\Delta \gg \gamma$*

**$U(x)$ = optical potential per photon = cavity frequency shift per atom**

$U(x) := \frac{\Delta_a}{\Delta_a^2 + \gamma_0^2} g_0^2 \cos^2(kx)$

**$\gamma(x)$ = photon loss per particle = radiation pressure photon**

$\gamma(x) := \frac{\gamma_0}{\Delta_a^2 + \gamma_0^2} g_0^2 \cos^2(kx)$

- **Dispersive limit $\Delta > \gamma$**
  - $U \gg \gamma$
  - $\Rightarrow$ atom-field interaction via **optical potential** $U$
  - $\Rightarrow$ **dipole force** dominates radiation pressure

- **Strong coupling re-defined**
  - $U \gg \kappa \gg \gamma$
  - $\Rightarrow$ single atom shifts cavity in or out of resonance
  - $\Rightarrow$ single photon creates an optical trap for an atom

- **Ultrastrong dispersive coupling**
  - $U \gg \Delta \omega_{mode} \gg \kappa$
  - $\Rightarrow$ single mode picture breaks down
  - $\Rightarrow$ nonlinear coupled multi mode model
**Point particle in optical resonator**

Field amplitude: $E = \exp\left[ -\kappa - \gamma(x) + i\Delta_c - iU(x) \right] E - \alpha$,  
Momentum: $\dot{p} = -|E|^2 \frac{d}{dx} U(x)$,  
Position: $\dot{x} = p/m$.

**Red detuning:**  
• Atoms drawn to field antinodes  
• Field gets maximal for atom at antinode

Particle moving in optical potential along axis

Lewenstein, PRL 95: ions  
Horak, PRL 97: atoms  
Vuletic, Chu, PRL 00: atoms  
Vitali, PRL 02: mirrors
**Semiclassical model:**
*Master equation for $\rho \Rightarrow PDE$ for Wigner function $\Rightarrow$* truncate at 2. order $\Rightarrow$ *Fokker Planck equation with drift+ diffusion $\Rightarrow$ equivalent Ito-stochastic differential equations*

\[
\begin{align*}
\frac{dx}{dt} &= \frac{p}{M}, \\
\frac{dp}{dt} &= -\hbar U_0 \left( \alpha_r^2 + \alpha_i^2 - \frac{1}{2} \right) \nabla f^2(x) dt + dP, \\
\frac{d\alpha_r}{dt} &= -\eta dt + (U_0 f^2(x) - \Delta_C) \alpha_i dt - (\kappa + \Gamma_0 f^2(x)) \alpha_r dt + dA_r, \\
\frac{d\alpha_i}{dt} &= - (U_0 f^2(x) - \Delta_C) \alpha_r dt - (\kappa + \Gamma_0 f^2(x)) \alpha_i dt + dA_i.
\end{align*}
\]

(quantum)noise forces $(dP, dA) \Rightarrow$ quantum expectation value $\Leftrightarrow$ stochastic average

Analytic solution for slow atoms for friction and diffusion

**friction**

\[
\overline{F_1} = -k^2 \eta^2 U_0^2 \frac{4\kappa^4}{4\kappa^4}
\]

**diffusion**

\[
\overline{D} = k^2 \kappa \eta^2 U_0^2 \frac{8\kappa^4}{8\kappa^4}
\]

\[
\begin{aligned}
\kappa_B T &= -\frac{\overline{D}}{\overline{F_1}} = \frac{\kappa}{2} \\
\frac{k_B T}{\kappa} &= \frac{\overline{D}}{\overline{F_1}} = \frac{\kappa^4}{6\kappa^4}
\end{aligned}
\]

**Cooling limited by cavity linewidth $\kappa$ !**

**Cooling time scale:**

\[
\tau_c = \frac{m}{2|\overline{F_1}|} = \frac{\kappa^4}{\eta^2 U_0^2 \omega_R}
\]
First Application: Cavity cooling

Theoretical simulation checked by analytical limits and QMC-Wavefunction Simulations

Temperature limit \( \sim h\kappa \)

Applications: trap + cool atoms, molecules, nanoparticles

Analogous: optomechanic cavity cooling of mirrors/membranes
**Few particle cavity cooling experiments in classical regime**

**Cavity cooling of a single atom**


Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, D-85748 Garching, Germany

- **first proof of principle experiment:**
  - MPQ München, Nature 2004

- **Ions:** MIT PRL 2009

- **Atomic ensembles:**
  - MIT, Vuletic PRL 2009

- More experiments by other groups: Meschede (EIT-cavity cooling), Stamper-Kurn, Barret, Baden, Barker, Renzoni, ???
- Temperature und cooling rate agree well with theory
- Ions: multiple vibrational modes addressed (Vuletic 2011), Barrett (2012),
Cavity cooling of an optically levitated nanoparticle

Kiesel, Blaser, Aspelmeyer, (PNAS 2014) ...

Trapping and efficient axial cooling

Cavity cooling of free silicon nanoparticles in high-vacuum

Asenbaum, Arndt, (2014) ...

radial and axial cooling in high vacuum

New results by K. Dholakia (St. Andrews), P. Barker (Imperial)
Scaling of resonator induced cooling

- Is resonator cooling good for larger ensembles?
- How far can we go in $T$?

Slow cooling for large ensembles!
(Scaling determined from numerics)
Quantum limit of cavity cooling: atomic motion + field quantized:

very good cavity with width smaller than recoil frequency (~\(\hbar k^2/2m\))

\[ \kappa < \omega_{recoil} \]

\[ H = \frac{p^2}{2m} + \frac{\hbar U_0 a^\dagger a \cos^2(kx)}{2} - \hbar \Delta E a^\dagger a - i\hbar \eta (a - a^\dagger) \]

quantum potential

\[ \dot{\rho} = \frac{1}{i\hbar} [H, \rho] + \kappa \left( 2\rho a a^\dagger - a^\dagger a \rho - \rho a a^\dagger \right) \]

**Strong pump:**

depth lattice

"blue" vibrational sideband of trapped atoms

**Weak pump:**

free motion with eff. mass

higher momentum states for a free gas

\[ a^\dagger B \]

\[ a^\dagger B^\dagger \]

\[ |m+1\rangle \]

\[ |m\rangle \]

\[ |m-1\rangle \]

vibrational trap states in cavity field

free space momentum states

Cavity mode tuned to
Experiment: cavity ground state cooling of „free“ atoms

„sub-recoil“ regime:
\[ \kappa < \omega_r \]
A. Hemmerich, Hamburg (Science 2012)

momentum distribution smaller than single photon recoil?
=> cooling towards degeneracy?

⇒ Cavity cooling with Bose stimulation to replace evaporation
⇒ BEC formation without particle loss

subrecoil width final distribution!
Cavity cooling dynamics and quantum statistics of two particles (can we ‘cavity‘-cool down to degeneracy from finite T)

„sub-recoil“ regime: \( \kappa < \omega_r \)

Groundstate cooling for \( \Delta \sim -4 \omega_r \)

Problem: nonlinear ladder higher momenta decoupled => very slow transfer
Efficiency of cooling stages depends on detuning and particle type (symmetric or antisymmetric wavefunction)
Example:

time evolution of optimized cooling sequences of two particles

„bosons“

„fermions“

Ring cavity much better in final stage
Cooling dynamics at optimal detuning for different particle quantum statistics

Ring cavity cools much better in the final stage towards degeneracy

subrecoil-cooling

best results with weak collisions
Momentum space pairing and quantum statistics: optimal cooling

- Momentum pair correlations
- Two bosons/distinguishable: correlations
- Two fermions: anticorrelations

- Quantum statistics changes particle momentum correlations
- Positive correlations enhanced in ring cavity and for bosons
Two bosonic particles in a ring cavity with dispersive interaction near motional ground state: momentum pairing

- Strongly pumped Cos-mode => mean field $\alpha$
- Scattered photons in Sin-mode => quantum operator $a$

Semiclassical classical point particle simulation

Start: random momenta

Final: correlated momenta

Center of mass momentum damps fast + anti correlated pairs decouple from dissipation

Relative motion: pairing by momentum anti-correlation
Quantum trajectory simulations (C++QED): two trapped bosons

Particles show positive correlations despite zero average center of mass momentum!

This is allowed by quantum mechanics for a momentum entangled state:

$$|\psi\rangle \propto |p, p\rangle \pm |-p, -p\rangle$$

How is this state forming??
Single trajectory analysis shows correlated quantum jumps of particles and field!

Quantum jumps (= photodetection) increase motional entanglement between the particles => strong conditional (heralded) entanglement induced by cavity dissipation / measurement
quantum trajectory simulations

conditional density matrix exhibits nonclassical correlations and entanglement

„squeezing“ of particle ( = mirror) distance!
Quantum simulations near $T\sim 0$

Particles started independent (= product state) in lattice ground state

diffusion and friction $\Rightarrow$ heating + correlations

Particle-particle entanglement (trace out field)

Figure 2: Momentum correlation. The parameters are the same as in figure 1 and $\max \langle p_1 p_2 \rangle / \sqrt{\langle p_1^2 \rangle \langle p_2^2 \rangle} \sim 0.08$. 
* Large entanglement in individual trajectories
* small average steady state entanglement
* very slow convergence of negativity with sample number

QMCWF-Simulations show slow convergence on correlations and entanglement!
intermediate summary:

* **Light forces in optical resonators:**
  * self trapping and cooling of all particles with sufficient polarizability
  * subrecoil cooling towards BEC’s and atom lasers
  * idealized implementation of optomechanics with ensemble
  * “infinite” range interactions perturb cooling and induce correlations and entanglement
**New:** Python wrapper ...

- **Goal:** simulation of composite open quantum systems
- **Methods of time evolution**
  - full master equation simulation
  - Monte Carlo wave-function simulation (MCWF, evolution with a non-hermitian Hamiltonian and random quantum jumps)
  - ensemble average of MCWF trajectories
- **Design objectives**
  - **Flexibility:** definition of composite systems using elementary free systems and interactions
  - **Performance:** e.g. maximal use of compile-time algorithms, adaptive stepsize, interaction picture
  - **Extensibility:** maximally reusable code, new elements can build on class hierarchy

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**Example of a composite system**

Two pumped lossy Qbits interacting with a mode of a cavity.

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**C++QED code base**

- Optimized ring cavity interaction
- Python interface

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**C++QED infrastructure**

- New cross-platform build system
- Improved support for Mac OS X / Windows
- Binary packages
- Python tool for convenient computing cluster integration
Ultracold gas near $T=0$

in a quantum optical lattice potential

Part II: Atomic dynamics in cavities beyond ‘mean field’
Reminder: Bose Hubbard model in fixed optical lattices

Theory:

Experiment:
Greiner et al. (2002) + many more.

Effective many body - Hamiltonian

\[ H = -J \sum_{\langle n,m \rangle} b^\dagger_n b_m + \frac{U}{2} \sum_n b^\dagger_n b_n (b^\dagger_n b_n - 1) + \sum_i (\varepsilon_n - \mu) b^\dagger_n b_n \]
Generalized Bose-Hubbard model in cavity generated fields

\[ H = \sum_{l=0,1} \hbar \omega_l a_l^\dagger a_l + \frac{1}{2} \frac{4\pi a_s \hbar^2}{m} \int d^3r \psi^\dagger(r) \psi^\dagger(r) \psi(r) \psi(r) + \int d^3r \psi^\dagger(r) H_0 \psi(r) \]

- Quantized light modes
- Nonlinear atom-atom interaction
- Adiabatically eliminated single-particle Hamiltonian

\[ H_0 = \frac{p^2}{2m_a} + V_{cl}(r) + \hbar g^2 \sum_{l,m=0,1} \frac{u_l^*(r) u_m(r) a_l^\dagger a_m}{\Delta_{ma}} \]

One-dimensional optical lattice: \( \mathbf{r}_m = x_m \mathbf{e}_x = m d \mathbf{e}_x \) for \( m = 1, 2, \ldots, M \)

Travelling wave cavities: \( u_{0,1}(\mathbf{r}_m) = \exp[i(mk_{0,1}d + \phi)] \)

Standing wave cavities: \( k_{0,1} = k_{0,1} \cos \theta_{0,1} \)
Bose Hubbard model for a single standing wave mode resonator

effective single atom Hamiltonian

\[
H_{\text{eff}} = \frac{p^2}{2m} + \cos^2(kx) \left( \hbar U_0 a^\dagger a + V_{cl} \right) - \hbar \Delta c a^\dagger a \\
- i\hbar \eta (a - a^\dagger) + \hbar \eta_{\text{eff}} h(y) \cos(kx) \left( a + a^\dagger \right)
\]
Hubbard model for a quantized single mode

\[ H = E_0 \hat{N} + E \hat{B} + (\hbar U_0 a^\dagger a + V_{cl}) \left( J_0 \hat{N} + J \hat{B} \right) \]

\[ - \hbar \Delta c a^\dagger a - i \hbar \eta (a - a^\dagger) + \frac{U}{2} \hat{C}. \]

\[ \hat{N} = \sum_k n_k = \sum_k b^\dagger_k b_k \]

\[ \hat{B} = \sum_k \left( b^\dagger_{k+1} b_k + h.c. \right) \]

Looks similar to standard Bose Hubbard model but parameters for lattice dynamics contain field operators (local and infinite range interactions)
single field mode as observable for atomic quantum statistics

Heisenberg equation for field amplitude operator $a$:

$$\dot{a} = \left\{ i \left[ \Delta_c - U_0 \left( J_0 \hat{N} + J \hat{B} \right) \right] - \kappa \right\} a + \eta$$

$$\hat{N} = \sum_k \hat{n}_k = \sum_k \hat{b}_k^{\dagger} \hat{b}_k$$

$$\hat{B} = \sum_k \left( \hat{b}_{k+1}^{\dagger} \hat{b}_k + h.c. \right)$$

atom number in cavity

local atom-atom coherence

field amplitude depends on quantum statistics and gets entangled with atomic distribution
Dynamical effects of a quantum potential:

bad cavity limit: effective Hamiltonian with eliminated field

\[ H = \left[ E + J \left( V_{cl} - \hbar U_0 \eta^2 \frac{\kappa^2 - \Delta_c^2}{(\kappa^2 + \Delta_c^2)^2} \right) \right] \hat{B} \]

\[ + 3 \hbar U_0 \eta^2 \Delta_c^\prime \frac{3 \kappa^2 - \Delta_c^2}{(\kappa^2 + \Delta_c^2)^2} J^2 \hat{B}^2 + \frac{U}{2} \sum_k \hat{n}_k (\hat{n}_k - 1) \]

rescaled hopping terms (sign change possible)

Nonlocal atom-atom interaction via nonlocal correlated hopping

Cavity parameters can be used to effectively tune size and type of interactions!
Thermodynamic limit and phases of cavity generated lattices

Cavity creates extra effective attraction or repulsion: bistable phases => phase superpositions of Mott + Superfluid in principle possible !?

M. Lewenstein, G. Morigi et. al. (PRL 2007, 2008)
Phase diagram in thermodynamic limit

Generalization to fermions, Morigi PRA 2008
full quantum dynamics beyond tight binding approximation: numerical solution for strong field and few particles by QMC - wavefunction simulations

[Graphs showing joint atom/photon probability and photon number distribution for weak and strong pump cases.]

efficient numerical studies of full system by quantum wave-function simulation framework C++QED available via (http://cppqed.sourceforge.net/) (Andras Vukics)
stationary solution zoo (two particles)

- atom field correlation and stationary entanglement!
- strong atom–atom correlations
- very strange states can form at least for few particles
Quantum dynamics of self-ordering in cavities

Modified-geometry: transverse pump: direct excitation of atoms from side!

Phase of excitation light depends on $x$-position

$$\dot{\sigma}_i = (i\Delta_A - \gamma)\sigma_i - g(z_i)a + \eta_x + \xi_A$$

$$\dot{a} = (i\Delta_C - \kappa)a + \sum_{i=1}^{N} g^*(z_i)\sigma_i + \xi_i$$

Field in cavity generated only by atoms

$R = 0$ for random atomic distribution

$R \sim Ng$ for regular lattice (Bragg)

see also: Vuletic, PRL 2001
Numerical simulations of coupled dynamics including atomic motion (classical point start with random distribution)

Phase diagram from stability analysis including diffusion

Particles spontaneously form crystalline order

(Niedenzu, EPL 2011)
Atom-field dynamics for very large particle number:  
=> Vlasov equation for particle distribution

Continuous density approximation for cold cloud: single particle distribution function

\[ f_s(x, p, t) := \frac{1}{N_s} \left\langle \sum_{j_s=1}^{N_s} \delta(x - x_{j_s}(t))\delta(p - p_{j_s}(t)) \right\rangle \]

\[ \Phi_s(x, \alpha) = \hbar U_{0,s}\left| \alpha \right|^2 \sin^2(kx) + \hbar \eta_s(\alpha + \alpha^*) \sin(kx) \]

Vlasov + field equation

\[ \frac{\partial f_s}{\partial t} + \frac{p}{m_s} \frac{\partial f_s}{\partial x} - \frac{\partial \Phi_s(x, \langle \alpha \rangle)}{\partial x} \frac{\partial f_s}{\partial p} = 0 \]

\[ \dot{\alpha} = (i \Delta_c - \kappa) \alpha - i \sum_s \int \left( \alpha U_{0,s} \sin^2(kx) + \eta_s \sin(kx) \right) f_s \, dx \, dp \]

Stability threshold of homogeneous distribution:

\[ \frac{N\eta^2}{k_B T} \int_{-\infty}^{\infty} g'(\xi) \, d\xi < \frac{\delta^2 + \kappa^2}{\hbar |\delta|} \]

Threshold at thermal equilibrium:

\[ U_{0,N} V_{opt} > \kappa^2 \]

- Frequency shift of cavity
- Pump laser opt. potential
- Cavity damping
Numerical simulation of Vlasov equation: cooling limit

periodic boundary conditions – 1 wavelength
time evolution of field intensity above threshold ($\sim \delta_c^2$)

negative detuning

positive detuning

instability confirmed but selfordering for negative detuning only!
special case here at CFEL: single side pumped ring (CARL /FEL)  Experiment : Zimmermann (Tübingen)

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x} \cdot \frac{\partial f}{\partial v} = 0 \]

\[ \int f(x, v, 0) d^3 x d^3 v = 1 \]

- density fluctuations backscatter light and get amplified
  => selfconsistent accelerated field
Simplest case: Instability in unidirectional cavity

\( \kappa t = 1 \)  
\( \kappa t = 3 \)  
\( \kappa t = 6 \)  
\( \kappa t = 11 \)  
\( \kappa t = 15 \)
Selforganization of a BEC at $T \sim 0$

\[
H = -\Delta_C a^\dagger a + \int_0^L \Psi^\dagger(x) \left[ -\frac{\hbar}{2m} \frac{d^2}{dx^2} + i\eta \cos kx(a^\dagger - a) \right] \Psi(x) dx,
\]

+ many more papers since

Two-mode approximation
=> Tavis-Cummings model

Nagy, Domokos, PRL (2010), NJP 2011
Fernandez-Vidal, Morigi PRA (2010)

- „Dicke Superradiant Phase“ transition
- Formation of a supersolid?

More refined theoretical models and numerical studies (tight binding):
Keldysh theory: Goldbart, Piazza, Strack, Zwerger, Diehl …
numerical studies: Vukics, Hofstetter, Bakhtiari, Thorwart,
Fermionic selfordering: Piazza, Keeling, …
Very simple toy model for dynamics: “decay of a quantum seesaw“

Two degrees of freedom: tilt angle $\phi$ and particle position $x$

Note: classical equilibrium point at $x=\phi=0$

but

product state of oscillator ground states is not stationary

field phase replaces tilt angle $\leftrightarrow$ occupation difference replaces position
Selfordering of trapped particles within a cavity: a numerical study

Flat box trap without prescribed lattice

- Cavity field Q-function
- Photons
- Photon variance
- No pump
- Weak pump
- Strong pump
- Particle density
- Several particle modes excited bi-modal Q-function of field

- Pump strength
Selfordering with multicolor pump field using several cavity modes: => competitive phase transitions

**Example: Box Potential**

\[ V(x) = \begin{cases} 
0 & x \in [-a,a] \\
\infty & \text{else} 
\end{cases} \]

\[ = \frac{1}{\sqrt{x-a}} \sin(K_i(x+a)) \quad x \in [-a,a] \]

\[ = \frac{\pi}{2a} i \equiv K_i \]

\[ E_i = \frac{\hbar^2}{2m} \left( \frac{K_i^2}{2a} \right) = \frac{\hbar^2}{2m} K_i^2 \]

Overlap-Integrals 18, 19:

\[ A_{nij} = \frac{1}{a} \int_{-a}^{a} \sin(K_i(x+a)) \sin(K_j(x+a)) \sin^2(kn(x+L)) dx \]

\[ B_{nij} = \frac{1}{a} \int_{-a}^{a} \sin(K_i(x+a)) \sin(K_j(x+a)) \sin(kn(x+L)) dx \]

Expand particle operators in trap eigenmodes

- \( H_{\text{particle}} \Psi_k(x) = E_k \Psi_k(x) \)
- Field operators: \( \hat{\Psi}(x) = \sum_k \Psi_k(x) \hat{c}_k \)

Nonlinear coupled oscillator model with tailorable coupling: pump amplitudes + detunings as control

Multimode Tavis Cummings model

\[ H = -\sum_n \Delta_n \hat{a}_n \hat{a}_n + \int dx \hat{\Psi}^\dagger(x) \left( \frac{-\Delta}{2m} + V(x) \right) \hat{\Psi}(x) \]

\[ + \int dx \hat{\Psi}^\dagger(x) \sum_n T_0 \omega_n \sin^2(k_n(x+L)) \hat{a}_n \hat{a}_n \hat{\Psi}(x) \]

\[ + \int dx \hat{\Psi}^\dagger(x) \sum_n \eta_n \sin(k_n(x+L)) (\hat{a}_n^\dagger + \hat{a}_n) \hat{\Psi}(x) \]

(„Hopfield model – associative memory“)
**BEC** - in a box with two pump frequencies

- **Pump 1 only**
  - Particle density
  - Superfluid
  - Field Q-functions
  - Vacuum

- **Both pumps**
  - Particle density
  - Many possible quasi-steady states!

- **No pumps**
  - Particle density
  - Vacuum

- **Pump 2 only**
  - Particle density
Single trajectory dynamics: particels and field jump between different configurations

formation and melting of order in time …. realistic scenarios far beyond our computing power …
Summary and Outlook

- Cavities can be used to confine, cool and control particle motion
- Replace evaporation to reach degeneracy and CW atom lasing
- Implement tailorable long range interactions
- Dynamic model of crystallisation of quantum systems
- Study associative memory and neural network models even in the quantum regime
Thanks for your attention!

Innsbruck University – visitors welcome!
How and when will selforganization happen here?

- pump creates optical lattice with atoms in lowest band
- cavity field from scattered lattice light

Effective Hubbard type Hamiltonian:

\[ H = \sum_{k,l} E_{k,l} b_k^\dagger b_l + \hbar U_0 \eta' g \sum_{k,l} J_{k,l} b_k^\dagger b_l + \hbar \eta' (a + a^\dagger) \sum_{k,l} \tilde{J}_{k,l} b_k^\dagger b_l - \hbar (\Delta_c - U_0) a^\dagger a \]

pump amplitude determined by atomic distribution operator

Multiparticle quantum description of selforganization in a lattice
Lowest energy states for atoms at two sites ...

\[ a = -i \frac{\eta'}{\kappa - i(\Delta c - U_0)} \tilde{J}_0 \left( b_1^\dagger b_1 - b_2^\dagger b_2 \right) \]

\[ a^\dagger a \sim \left( b_1^\dagger b_1 - b_2^\dagger b_2 \right)^2 \]

\[ \frac{1}{\sqrt{2}} \left( |\text{left}\rangle |\alpha\rangle \pm |\text{right}\rangle |-\alpha\rangle \right) \]

... show atom field entanglement

- Note: strongly entangled state
- Symmetry leads to zero field but nonzero intensity (photons)
- How does entanglement and intensity grow?
real Cavity QED = open system

input and output channels => damping + fluctuations + decoherence

but

allow measurement + (feedback) control

particles scatter spontaneously at rate $\gamma$

$\alpha$ -> Photon escape through mirror at rate $\kappa$

Cavity QED (=strong coupling) limit, if $\omega_p\omega_a >> g >> (\kappa,\gamma)$

Experiments in wide range:

- Microwave cavity $\omega \sim 10^9$ Hz
- Optical cavity $\omega \sim 10^{14}$ Hz

Consequences:

- Nonlinear atomic response at less than a single photon: $n_0$
- Single atom shifts cavity by more than a linewidth $N_0$

Gedankenexperiments of Quantum Mechanics realized:

e.g. Haroche, Walther, Kimble, Rempe, … + many more recently (circuit QED)
**Trapped particle in a ring cavity with symmetric pump**

\[ E(x,t) \sim a_c \cos(kx) + a_s \sin(kx) \]

**point particle motion:**

*single mode*

*two modes = ring cavity*

Efficient trapping and cooling towards very low velocities!

*Gangl, PRA 99, Exp. by Hemmerich (Hamburg) and Zimmermann (Tübingen)*
trapping and cooling to the quantum limit in a ring cavity

ultracold + localized particles =>
atom-field Hamiltonian for quantized motion:

\[
H = \frac{\hat{p}^2}{2m} - \hbar \Delta \left( a_c^\dagger a_c + a_s^\dagger a_s \right) - \hbar U(\hat{x}) + i \hbar \left( \eta a_c^\dagger - \eta^* a_c \right)
\]

\[
U(\hat{x}) = a_c^\dagger a_c U_c(\hat{x}) + a_s^\dagger a_s U_s(\hat{x}) + (a_c^\dagger a_s + a_s^\dagger a_c) U_{cs}(\hat{x})
\]

E(x,t) ~ a_c \cos(kx) + a_s \sin(kx)

generic setup:
strong pump of cosine mode:
=> deep trap for particle
=> mode in coherent state $\alpha$

\[
\omega_m^2 = 4\omega_R U_0 \alpha^2
\]

linear coupling $'(a_s^\dagger + a_s) x'$
=> “optomechanical cooling”

quadratic coupling $'a_c^\dagger a_c \ x^2'$
=> trap + $x^2$ nonlinearity
Selfordering via a continuum of light modes

trapped particles interact by collective scattering and dipole-dipole exchange

Example: dipole-dipole interaction enhanced by optical fiber or microstructure

A. Rauschenbeutel

\[
\frac{\partial^2 E}{\partial z^2} + \left(\beta_m^2 + k_L^2 \tilde{\chi}\right) E = -k_L^2 \tilde{\chi} E_L, \tag{1a}
\]

\[
\frac{\partial f}{\partial t} + \frac{p_z}{m} \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \left(U - \alpha |E|^2 + 2E_L E_r\right) \frac{\partial f}{\partial p_z} = 0
\]

- particles order to confine light and themselves
- long range phononlike excitations
- self optimized light harvesting structure

T. Griesser, PRL 2013

(see also: Chang et.al, PRL 2013)
Analytic model in the linear regime:

Dynamic equations:

\[
\frac{d}{dt} \langle a^{\dagger} a \rangle = -2 \kappa \langle a^{\dagger} a \rangle - i \tilde{U}_0 \langle a Q \rangle - \langle a^{\dagger} Q \rangle
\]

\[
\frac{d}{dt} \langle Q^2 \rangle = \omega_m \langle A \rangle
\]

\[
\frac{d}{dt} \langle A \rangle = 2 \omega_m \langle P^2 \rangle - \langle Q^2 \rangle + 2 \tilde{U}_0 \left( \langle a Q \rangle + \langle a^{\dagger} Q \rangle \right)
\]

\[
\frac{d}{dt} \langle P^2 \rangle = -\omega_m \langle A \rangle + 2 \tilde{U}_0 \left( \langle a P \rangle + \langle a^{\dagger} P \rangle \right)
\]

\[
\frac{d}{dt} \langle a Q \rangle = \omega_m \langle a P \rangle - (\kappa - i \Delta) \langle a Q \rangle + i \tilde{U}_0 \langle Q^2 \rangle
\]

\[
\frac{d}{dt} \langle a P \rangle = (-\kappa + i \Delta) \langle a P \rangle - \omega_m \langle a Q \rangle + \tilde{U}_0 \left( \langle a^{\dagger} a \rangle + 1/2 + \langle a^2 \rangle + \frac{i}{2} \langle A \rangle \right)
\]

\[
\frac{d}{dt} \langle a^2 \rangle = -2(\kappa - i \Delta) \langle a^2 \rangle + 2i \tilde{U}_0 \langle a Q \rangle,
\]

Steady states:

\[
\langle a^{\dagger} a \rangle = -\frac{U_0^2 (\Delta^2 + \kappa^2)}{4 \omega_m \Delta (\kappa^2 + \Delta^2) + 8 \tilde{U}_0^2 \Delta^2},
\]

\[
\langle Q^2 \rangle = -\frac{(\kappa^2 + \omega_m^2 + \Delta^2)(\kappa^2 + \Delta^2) + 2 \tilde{U}_0^2 \omega_m \Delta}{4 \omega_m \Delta (\kappa^2 + \Delta^2) + 8 \tilde{U}_0^2 \Delta^2}
\]

\[
\langle P^2 \rangle = -\frac{(\kappa^2 + \omega_m^2 + \Delta^2 + 2 \tilde{U}_0^2 \Delta/\omega_m)(\kappa^2 + \Delta^2) + 2 \tilde{U}_0^2 \omega_m \Delta}{4 \omega_m \Delta (\kappa^2 + \Delta^2) + 8 \tilde{U}_0^2 \Delta^2}
\]

Examples:

* Ground-state cooling !
* acceleration by strong power !
Numerical Monte Carlo wave function simulations beyond linearized potential (C++QED-package)

Single quantum particle in a ring cavity with dispersive interaction: frequency dependence of cooling

Particle energy

cooking rate

ground state cooling when mode is tuned the antistokes line: $\Delta \sim \omega >> \kappa$
any polarizable, nonabsorptive particle can be cooled with sufficient power!

(-> Romero-Isart)

(new proposed experiments with beads: P. Barker, M. Raizen, M. Aspelmeyer, J. Kimble)
Single trajectory analysis of groundstate cooling:
Quantum jumps of particle and field near ground state!

- sin-mode photon number
- particle state position uncertainty
- particle state occupation number

Particle jumps between two lowest states (parity change !)
Two particles in a ring cavity with dispersive interaction: near their ground state: $T \sim 0$

classical point particles

* Strongly pumped $\text{Cos}(kx)$ mode $\Rightarrow$ mean field $\alpha$
* Scattered photons in $\text{Sin}(kx)$ mode $\Rightarrow$ quantum operator $a$
* two particles : classical description of motion
Dynamical effects of a quantum potential:

\[ H = \left[ E + J \left( V_{cl} - \hbar U_0 \eta^2 \frac{\kappa^2 - \Delta_c^2}{(\kappa^2 + \Delta_c^2)^2} \right) \right] \hat{B} \]  
\[ + 3\hbar U_0^2 \eta^2 \Delta_c \frac{3\kappa^2 - \Delta_c^2}{(\kappa^2 + \Delta_c^2)^2} J^2 \hat{B}^2 + \frac{U}{2} \sum_k \hat{n}_k (\hat{n}_k - 1) \]  

bad cavity limit: effective Hamiltonian with eliminated field

Cavity parameters can be used to effectively tune size and type of interactions!
Simulated single atom dynamics for two wells: photon-assisted or photon blocked tunneling

at \( t=0 \) atom prepared at right well:

- jumps in photon number + atomic state
- effective model contains weighted average of tunnel amplitudes
Generalization to multimode confocal cavity:

S. Gopalakrishnan, B. L. Lev, P. M. Goldbart

„Quantum Brazovskii transition“

P. Strack and S. Sachdev

- Dicke quantum spin glass of atoms and photons
- Exploring models of associative memory via cavity quantum electrodynamics
Numerical simulations of coupled dynamics including atomic motion (start with random distribution at Doppler temperature)

24 atoms

Pump strength vs number of photons

Photon number

Number of atoms vs Mandistance from antinodes

$\text{n} \sim N^2$

clear threshold!

Superradiance!

Selforganization!

quadratic dependence of cavity photons on atom number

$\Rightarrow$ stimulated emission dominates over spontaneous emission for large $N$

$\Rightarrow$ internal state unchanged (no repumper required)
diffusion and temperature: kinetic equation for fluctuations

\[
\frac{\partial \langle f_l \rangle}{\partial t} + v \frac{\partial \langle f_l \rangle}{\partial x} - \frac{1}{m_l} \frac{\partial \langle \Phi_l \rangle}{\partial x} \frac{\partial \langle f_l \rangle}{\partial v} = \frac{1}{m_l} \left\langle \frac{\partial \delta \Phi_l}{\partial x} \frac{\partial \delta f_l}{\partial v} \right\rangle
\]

Below self consistent threshold:

\[\langle F(v) \rangle \propto \left(1 - (1 - q) \frac{mv^2}{2k_BT}\right)^{\frac{1}{1-q}}\]

\[k_BT = \hbar \frac{\kappa^2 + \delta^2}{4|\delta|} = \frac{\hbar \kappa}{2}\]

stationary velocity distribution

Above self consistent threshold:

\[N |U_0| V_{opt} \propto \left(\frac{\kappa^2 + \delta^2}{2|\delta|}\right)^2 \frac{2}{3-q} \delta = -\kappa \frac{2}{3-q} k^2\]

time evolution of 'hot' ensemble
spatial average: kinetic equation for velocity distribution

\[ \frac{\partial}{\partial t} \langle F \rangle + \frac{\partial}{\partial v} \left( A[\langle F \rangle] \langle F \rangle \right) = \frac{\partial}{\partial v} \left( B[\langle F \rangle] \frac{\partial}{\partial v} \langle F \rangle \right) \]

\[ A[\langle F \rangle] := \frac{2\hbar k \delta \kappa \eta^2}{m} \frac{k v}{|D(i k v)|^2} \]
\[ B[\langle F \rangle] := \frac{\hbar^2 k^2 \eta^2 \kappa \kappa^2 + \delta^2 + k^2 v^2}{2m^2} \frac{|D(i k v)|^2}{|D(i k v)|^2} \]

numerical solution:

selforganization time

very weak particle number dependence on selforganization time + cooling!
Eperiment ETH:
Observation of the phase transition to new phase with coherence + ordering present ("supersolid phase")

\[ \Psi(x) = \frac{1}{\sqrt{L}} c_0 + \sqrt{\frac{2}{L}} c_1 \cos kx \]

Implementation of „Dicke Superradiant Phase“ transition

Measurement of phase diagram:

in ordered region:
coherence + ordering present: “supersolid phase”

FIG. 40 Phase diagram of the Dicke model, from (Baumann et al., 2010).
Two atoms measures ordering:

\[ |\Psi(t)\rangle = (|-, -2\alpha\rangle + |+, 2\alpha\rangle) \]

\[ |\Psi'(t)\rangle \propto a |\Psi(t)\rangle \propto |-, -2\alpha\rangle - |+, 2\alpha\rangle \]

\[ \rightarrow = [(2,0) - (0,2)] \text{ state does not tunnel!} \]

\[ 1 \text{ „Mott“-insulator (1,1)} \]

0 Ordered states \{(2,0) +/- (0,2)\}

„spontaneous“ ordering via photon scattering by sign change

Bound pairs (Bose Hubbard)

tilted potential
Part III) Quantum dynamics and controlled interactions of few particles in multimode “cavities

- cosine-mode as trap
- sine–mode for cooling + control

\[
\text{atom phase locks field modes and changes intensity distribution} \\
\Rightarrow \quad \text{atom drags node along ist path to stay at intensity maximum = field minimum}
\]
Quantum model in a ring cavity

atom-field Hamiltonian for quantized motion:

\[
H = \frac{\hat{p}^2}{2m} - \hbar \Delta \left( a_c^* a_c + a_s^* a_s \right) - \hbar U(\hat{x}) + i\hbar \left( \eta a_c^* - \eta^* a_c \right)
\]

\[
U(\hat{x}) = a_c^* a_c U_c(\hat{x}) + a_s^* a_s U_s(\hat{x}) + (a_c^* a_s + a_s a_c^*) U_{cs}(\hat{x})
\]

**pumped cosine mode:**

⇒ mode in coherent state \( \alpha \)

⇒ deep harmonic trap for particle

**unpumped sine mode:**

⇒ mode near vacuum state

⇒ linear coupling

⇒ cooling + measurement

**deep trap limit:**

\[
H = \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_m^2 \hat{x}^2 \right] - \hbar \Delta a_c^* a_c - \hbar U'_0 (a_s + a_s^*) \hat{x}
\]

- **quadratic coupling** \( a_c^* a_c x^2 \)

⇒ trap + \( x^2 \) nonlinearity

- **linear coupling** \((a_s^* + a_s) x\)

⇒ “optomechanical cooling”
**CW- operation of atom-photon pair laser:**

\[ \Rightarrow add \ CW \ ‘hot’ \ atom \ source \ or \ incoherent \ pump \]

stimulated amplification of light and atomic ground state population via blue Raman sideband

\[ \mathcal{H}_{int} = \eta (a_d b_c^\dagger c + a b c_d^\dagger) \]

T. Salzburger et. al. , PRL
Cold gas in a quantum optical potential (mean field limit)

* Cavity field generates dynamical optical lattice with quantum properties
* Atoms act on the cavity field depending on their quantum state

Correlated / entangled dynamics of field amplitude and particle wave-function

Coupled Maxwell - Schrödinger equation
Cold dilute gas in a cavity generated optical potential at finite $T$ (Vlasov mean field approach)

Continuous density approximation for cold cloud: dynamic refractive index

$$f(x, v, t) = \frac{m}{2N\pi\hbar} \int e^{-izmv/\hbar} \rho_{p,1}\left(x + \frac{z}{2}, x - \frac{z}{2}, t\right) dz$$

Kinetic limit-Vlasov equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{U_0|\alpha|^2}{2} \sin(2kx)\left(f(x, v + v_R) - f(x, v - v_R)\right) = 0$$

field dynamics:

$$\dot{\alpha} = [-\kappa + i(\Delta_c - NU_0/2)]\alpha - i\frac{NU_0}{2}\alpha \int_{-\infty}^{\infty} dv \int_{0}^{\lambda} f(x, v, t) \cos(2kx) dx + \eta$$
Stationary solution for strong field and many particles: bistability

- **moderate pump**
- **strong pump**

**Two stable solutions:** nonlinear optics

(a) weak field and homogeneous density
   \[ (kT >> V_{\text{opt}}) \]
(b) strong field and modulated density
   \[ (kT << V_{\text{opt}}) \]

- **no stable solution:** limit cycle
unstable regime for strong pump:

dynamic solution shows density waves with limit cycle behaviour

related experiments:
A. Hemmerich (transverse motion)
J. Eschner (thermal cloud)
Quantum dynamics of many particles and field near $T \sim 0$

BEC in optical lattice with dynamic (quantum) properties

Mean field description of many particles and field

Gross-Pitajevski $\Leftrightarrow$ Maxwell

\[
\frac{d}{dt} \alpha(t) = [i \Delta_c - i N \langle \hat{U}(\hat{x}) \rangle - \kappa] \alpha(t) + \eta, \quad (1a)
\]

\[
\frac{d}{dt} \psi(x,t) = \left\{ \frac{\hat{P}^2}{2m} + |\alpha(t)|^2 U(x) + N g_{coll} |\psi(x,t)|^2 \right\} \psi(x,t).
\]

coupled nonlinear and nonlocal equations with a wealth of dynamic effects

Refs: Horak, Barnett, Hammerer, Zoller, Meystre, Liu, Bhattacherjee...

Experiments: Esslinger, Reichel, Zimmermann, Hemmerich, Stamper-Kurn, Vuletic, Treutlein...
BEC in a cavity: quantum T=0 limit

two relevant momentum modes:
1. homogeneous state
2. diffracted wave: cos(2kx)

\[ \psi(x, t) = c_0(t) + c_2(t) \sqrt{2} \cos(2kx) \]

two coupled oscillators

\[ X = 2 \sqrt{\frac{1}{N}} \text{Re}(c_0^* c_2) \]

⇒ optomechanics – Hamiltonian at T=0

\[ \ddot{X} + (4 \omega_{rec})^2 X = -\omega_{rec} U_0 \sqrt{8N} \langle \hat{a}^\dagger \hat{a} \rangle \]

mirror position = amplitude of density wave

BEC parameter regime:
- start at ground state (T=0)
- strong single photon coupling
- atoms and light can be measured
- nonlinear regime: bistable response

ideal „optomechanics“ test toolbox ⇒ next talk!
Dynamical Coupling between a Bose-Einstein Condensate and a Cavity Optical Lattice

Stephan Ritter\textsuperscript{1,2}, Ferdinand Brennecke\textsuperscript{1}, Christine Guerlin\textsuperscript{1}, Kristian Baumann\textsuperscript{1}, Tobias Donner\textsuperscript{1,3}, Tilman Esslinger\textsuperscript{1}\textsuperscript{*}

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(Dated: November 24, 2008)

* two mode BEC expansion works very well
* study of zero T optomechanics in the instable regime
* nonlinear oscillations + atom field entanglement
Cavity cooling of molecules or heavier particles

Examples: Cavity of $V=1 \text{ mm}^3$ at $1.5 \mu \text{ laser}$

| Particle | $m$ (amu) | $\chi (\text{Å}^3 \times 4\pi \varepsilon_0)$ | $\sigma_a (\text{Å}^2)$ | $\omega_r$ (MHz) | $|U_0|$ (MHz) | $2\gamma_a$ (MHz) | $2\gamma_s$ (MHz) |
|----------|----------|---------------------------------|-----------------|-----------------|----------------|-----------------|-----------------|
| Li       | 7        | 24                              |                 | $8.0 \times 10^{-2}$ | $1.9 \times 10^{-9}$ |                 | $8.9 \times 10^{-18}$ |
| C$_{60}$ | 720      | 83                              | $\sim 10^{-4}$  | $7.7 \times 10^{-4}$ | $6.5 \times 10^{-9}$ | $\sim 10^{-12}$ | $1.0 \times 10^{-16}$ |
| He$_{1000}$ | 4000 | 200                             |                 | $1.4 \times 10^{-4}$ | $1.6 \times 10^{-8}$ |                 | $6.2 \times 10^{-16}$ |
| Li$_{1000}$ | 7000 | 5501                            | $2.6 \times 10^{-1}$ | $8.0 \times 10^{-5}$ | $4.3 \times 10^{-7}$ | $1.5 \times 10^{-8}$ | $4.7 \times 10^{-13}$ |
| (SiO$_2$)$_{1000}$ | 60000 | 2901                           | $6.1 \times 10^{-11}$ | $2.0 \times 10^{-5}$ | $2.3 \times 10^{-7}$ | $3.7 \times 10^{-18}$ | $1.3 \times 10^{-13}$ |
| Au$_{1000}$ | 197000 | 4180                           | $8.2 \times 10^{-2}$ | $2.8 \times 10^{-6}$ | $3.3 \times 10^{-7}$ | $4.8 \times 10^{-9}$ | $2.7 \times 10^{-13}$ |

- **+** single setup for wide range of species
- **-** but need to prepare useful initial confinement in mode
- **-** Watts of pump power at finesse $F > 10^4$

Note: speed up cooling for weak coupling by more power!
(S. Nimmrichter, NJP 12, 2010, Salzburger 2009, Deachapunya EPJD 08)