Ultracold Quantum Gases
Part 3: Artificial gauge potentials

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Ultracold Quantum Gases

3.5 Engineering and probing topological band structures

• Topology and material properties
  » Energy bands of a solid can be topologically non-trivial
    • Berry curvature, Chern number
  » Topologically non trivial bands give rise to new properties
    • Anomalous velocity
    • Quantized conductance
    • Topological insulators
    • Edge states

• Quantum gases give access to so far non-measurable quantities
  » Engineering of topologically non-trivial band structures
  » Measurement of topological properties (Berry curvature)
  » Direct observation of edge states
Topological properties of Bloch bands

- Berry curvature in Bloch bands
  - Eigenstates $|u_{n,k}\rangle$ (band index $n$, quasi-momentum $k$)
  - Berry connection $A_n(k) = i\hbar \left( u_{n,k} | \nabla_k u_{n,k} \right)$
  - Berry curvature $B_n(k) = \nabla_k \times A_n(k)$
  - Magnetic field in momentum space $F \propto -\dot{k} \times B_\lambda(k)$

- Berry phase: Geometrical phase accumulated around a closed path
  $\Phi_{n,\text{geom}} = \frac{e}{\hbar} \oint A_n(k) \cdot dk = \frac{e}{\hbar} \iint B_n(k) \cdot dS$

- Chern number
  - Integral of the Berry curvature over the Brillouin zone
    $C_n = \frac{1}{2\pi} \iint_{FBZ} B_n(k) \cdot dk$
  - Gauge invariant

"The remarkable and rather mysterious result of this paper..." Berry 1984
Non-trivial topological bands and material properties

• Topological insulator
  » Electronic band structure: band insulator with Fermi level falling between valence and conduction band
  » Insulating in the bulk
  » Metallic at the surface: edge/surface-states (bulk energy gap)

• Chern number and transport properties
  » Influences the transport properties
    • Anomalous velocity
    • Quantized conductance (Quantum Hall effect)
  » Determines the number of edge states (Bulk/Edge equivalence)
Chern number and transport measurements

• Band velocity in a 1D lattice
  » Atom cloud submitted to a constant force $F$ along $y$
  » Average velocity of the eigenstate $|u_{n,k}\rangle$

$$v_n(k) = \langle u_{n,k} | \hat{v} | u_{n,k} \rangle = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k}$$

» Semi-classical equations of motion for a wave-packet

$$\hbar \dot{x}_c = \hbar v_n(k_c) = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k}$$

$$\hbar \dot{k}_c = F$$

• Bloch oscillations
Chern number and transport measurements

• Anomalous velocity in 2D lattices
  » Modification of the velocity along the transverse direction $x$
  \[
  v^x_n(k) = \left\langle u_{n,k} \left| \hat{v}^x \right| u_{n,k} \right\rangle = \frac{1}{\hbar} \frac{\partial E_n(k)}{\partial k_x} - \frac{F_y}{\hbar} B_n(k)
  \]
  \[
  B_n(k) = i \left( \left\langle \partial_{k_x} u_{n,k} \left| \partial_{k_y} u_{n,k} \right\rangle \right) - \left\langle \partial_{k_y} u_{n,k} \left| \partial_{k_x} u_{n,k} \right\rangle \right)
  \]
  $1^{st}$ term: Usual band velocity responsible for Bloch oscillations
  $2^{nd}$ term: Anomalous velocity due to the Berry curvature
  » Net drift transverse to the applied force

• Transverse velocity for uniformly populated bands
  » Number of states per band $N_{\text{states}} = A_{\text{sys}} / A_{\text{cell}}$
  » Average particle number uniform over the Brillouin zone $\rho^n(k) = \rho^n = N^n / N_{\text{states}}$
  » Mean transverse velocity
  \[
  v^x_{\text{tot}} = \sum_n \rho^n \sum_k v^x_n(k) \rightarrow - \frac{F_y A_{\text{cell}}}{\hbar} \sum_n C_n \quad \left( \int_{BZ} \frac{\partial E_n(k)}{\partial k_x} d^2k = 0 \right)
  \]
Quantum Hall effect

- Hall effect
  - 2D electrons gas in presence of a magnetic field
  - Electrons are deviated by the Lorentz force
  - Separation of charges induces an electric field
  - Hall voltage non-zero

- Quantum Hall effect – macroscopic occupation of Landau levels
  - Macroscopic degeneracy of each level (sample area $A$)
    
    $$ p = \frac{A}{2\pi l_{mag}^2} = \frac{eAB}{2\pi\hbar} = \frac{\Phi}{\Phi_0} = N\phi $$

  - Filling factor – number of Landau levels involved $\nu = \frac{N_e}{N\phi}$

  - Quantum Hall effect reached for $\nu = 1$

  - Effect of the chemical potential
    Insulating material when chemical potential between a filled and an empty band

Gross and Marx, Festkörperphysik (chapter 10)
Quantum Hall effect

• Integer Quantum Hall effect
  » Current along x fixed \( J_x = n_{2D} e \nu_x \)
  » Gate voltage \( U_g \) varied in order to vary \( \mu \)
  » Measurement of \( U_x \) and \( U_y \)
    \[ U_x = \rho_{xx} L_x J_x \]
    \[ U_y = \rho_{xy} L_y J_x \]

• Observations
  » \( U_x \) vanishes periodically:
    Insulating when \( \mu \) between 2 Landau levels
    \[ \rho_{xx} = \rho_{yy} = 0 \]
  » \( U_y \) has plateaus for the same values of \( U_g \)

Completely unexpected...

Von Klitzing constant: \( R_K = \frac{e}{\hbar^2} \)

Effect of non-trivial topological bands!
Quantum Hall effect

- 2D polarized Fermi gas at $T=0K$
  - Fermi energy within a spectral gap
  - Perfect filling of the bands below the gap
  
  $$\rho = \frac{N_{tot}}{N_{states}} = 1 \text{ for } E_n < E_F$$

- Quantum Hall effect
  - Total transverse velocity
    
    $$v_{tot}^x = -\frac{F_y A_{sys}}{\hbar} \sum_{E_n < E_F} C_n$$
  
  - Electric Hall conductivity $\sigma_{x,y}$
    
    $$j_x = \sigma_{x,y} E_y$$
    
    $$j_x = \frac{e v_{tot}^x}{A_{sys}} \Rightarrow \sigma_{x,y} = \frac{e^2}{\hbar} \sum_{E_n < E_F} C_n$$

  - Transport measurements reveal the Chern numbers

\[\text{Gross and Marx, Festkörperphysik (chapter 10)}\]
Quantum Hall effect

• Integer Quantum Hall effect
  » Quantized conductivity
  » Transport measurements reveal topological properties

• Fractional Quantum Hall effect
  » Plateaus at fractional values of the Hall resistance
  » Collective behavior: condensation of the electron gas
  » Microscopic origin unknown
  Induced by e-e repulsion?
  Quantum simulation with model systems!
Engineering topologically non-trivial models - Harper Model

• Uniform magnetic field on a lattice
  » Energy spectrum - Hofstadter butterfly
  » For rational values of the flux \( \alpha = \Phi / \Phi_0 = p' / p \)
    • Increased spatial periodicity \( p a \) (magnetic cell)
    • The energy band for \( \Phi = 0 \) splits into \( p \) sub-bands
    • Each sub-band has a non-zero Chern number

• Consequence of the non-trivial topology
  » Quantized conductance: quantum Hall effect
  » Edge states: macroscopic consequence of the cyclotron orbits
    induced by a magnetic field truncated at the sample’s boundary

• Realized for quantum gases
  Periodic amplitude modulation in a square lattice

\[ \alpha = \Phi / \Phi_0 = p' / p \]

Engineering topologically non-trivial models - Haldane model

• Graphene-like honeycomb lattice
  » Unit cell contains two equivalent sites A and B
  » Nearest neighbor tunneling of amplitude $J$ ($A \leftrightarrow B$)
  » Band structure: two bands touching at the Dirac points
  » Berry curvature non zero
    • Berry phase around the Dirac point
      • $B_1(k) = -B_1(-k) \Rightarrow C_1 = 0$

• Breaking time-reversal symmetry
  » Addition of complex next-neighbor tunneling ($A \leftrightarrow A$ or $B \leftrightarrow B$)
  » Lift the degeneracy at the Dirac points
  » 2 sub-bands separated by a gap with Chern numbers +1 and -1
    $B_1(k) \neq -B_1(-k) \Rightarrow C_1 \neq 0$

• Realized for quantum gases
  Circular acceleration of an honeycomb lattice

Esslinger, Nature 515, 238 (2014)
Evidencing topological properties with quantum gases

• Chern number
  » Transport measurements
    • Anomalous velocity
    • Quantized conductance
  » Counting the edge states

• Measuring the Berry phase with a momentum space interferometer

• Mapping the Berry curvature
Mapping the Berry curvature

Sengstock, Weitenberg, Science 352, 1091 (2016)
Mapping the Berry curvature

- Mapping the Berry curvature
  - Obtain the eigenstates in a k-dependent manner

Sengstock, Weitenberg, Science 352, 1091 (2016)
Measuring the Berry phase

• Honeycomb lattice
  » Berry curvature non zero
  » Berry phase of $\pi$ accumulated around the Dirac points
  » Opposite signs for the 2 Dirac points

• Measurement of the Berry phase
  » Berry flux analog to a magnetic flux
  » Aharonov-Bohm interferometer: observation of the phase accumulated proportional to the magnetic flux
  » Berry flux interferometer: closed path in reciprocal space

Direct observation of edge states

• Edge states
  » Metallic states located at the edge of the sample
  » Reveal non-trivial bulk properties (topological insulators)
  » For non-interacting fermionic system:
    Number of edge states = Chern number of the filled bands

• Edge states for quantum gases
  » Requires a non-zero Chern number for the lowest band
  » Harper model: macroscopic consequence of the cyclotron orbits induced by a magnetic field truncated at the physical boundary of the sample
  » Direct observation challenging
    • Corresponding to mass current (Time-of-Flight imaging)
    • Large imbalance between population bulk and edge states
    • Difficult to observe in harmonic traps (no sharp edges): box potentials required
Direct observation of edge states

• Synthetic magnetic fields in synthetic dimensions
  » Magnetic fields are two-dimensional objects
  » Synthetic magnetic field
    • One dimensional lattice with tunneling $J$
    • Extra dimension: internal degree of freedom (nuclear spin)
    • Two-photon Raman transition couples the spins and induces a complex tunneling amplitude along the extra dimension
  » Realization of the Harper model

• Direct observation of edge states
  » Two-legs ladder with fermions
  » Opposite mass currents along the two legs
  » Chiral dynamics revealed
    by spin resolved time-of-flight measurement
    \[ h(k) = n(k) - n(-k) \]

Spielman & Inguscio, Science 349 (2015)